Slow Crack Propagation in Heterogeneous Materials

J. Kierfeld[1] and V. M. Vinokur[2]

[1]*Max Planck Institute of Colloids and Interfaces, Science Park Golm, 14424 Potsdam, Germany*
[2]*Materials Science Division, Argonne National Laboratory, 9700 South Cass Avenue, Argonne, Illinois 60439, USA*

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Statistics and thermally activated dynamics of crack nucleation and propagation in a two-dimensional heterogeneous material containing *quenched randomly distributed* defects are studied theoretically. Using the generalized Griffith criterion we derive the equation of motion for the crack tip position accounting for dissipation, thermal noise, and the random forces arising from the defects. We find that aggregations of defects generating long-range interaction forces (e.g., clouds of dislocations) lead to anomalously slow creep of the crack tip or even to its complete arrest. We demonstrate that heterogeneous materials with frozen defects contain a large number of arrested microcracks and that their fracture toughness is enhanced to the experimentally accessible time scales.

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Fracture mechanisms and their relation to the material structure is a long-standing problem [1]. Ideal crystals are subject to fast brittle fracture—as was first explained by Griffith [2]—while homogeneously amorphous systems exhibit slow ductile fracture controlled by the plastic deformation at the crack tip [1,3]. Real materials are neither of the above. Real crystals do contain defects; but even in the ultimately disordered substances the defects are not distributed homogeneously but form spatially inhomogeneous aggregates such as inclusion clusters and/or dislocation pileups [1,4,5]. It is intuitively plausible that defect aggregates promote crack nucleation as the crack can settle at an energetically favorable nucleation site. At the same time one can expect that random heterogeneities impede the subsequent crack propagation process as shown in Fig. 1. This poses the important question about the ultimate effect of frozen inhomogeneities on the fracture mechanism and, in particular, whether frozen defects enhance or degrade the fracture toughness of a material.

The inspiring work [5] discussed the disorder-stimulated *nucleation* of critical cracks. In this Letter, we focus on the *dynamics* of cracks in heterogeneous materials and investigate fracture probabilities and the statistics of the fracture times. We consider *both* zero- and finite-temperature crack dynamics, the latter being governed by thermal activation. We restrict ourselves to the simplest case of cracks in a thin (quasi-) two-dimensional (2D) ideally elastic plate containing random heterogeneities. In two dimensions a crack front is a point—the crack tip; thus the additional effects arising from crack front roughening are absent. Building on Griffith's concept of energy balance [2], we consider crack tip motion governed by the dynamic energy release rate [3,6] and derive the equation of motion for the crack tip. We include both dissipative and thermal forces, and the position-dependent random forces acting on the crack tip due to frozen material defects. We discuss three basic kinds of frozen inhomogeneities: (i) bond strength variations, (ii) random impurities resulting in local compression of the elastic medium, and (iii) frozen dislocations interacting with the crack. The range of elastic interactions with the crack tip increases when going from the type (i), to types (ii) and (iii).

We find that the supercritical cracks can be *arrested* by heterogeneities inducing long-range elastic forces, i.e., by frozen dislocations (iii). We show that thermally activated cracks exhibit *anomalously slow dynamics* with vanishing mean velocity for all three types of disorder. We conclude that quenched defects effectively slow down the crack propagation and derive experimentally observable characteristic material properties such as the statistics of the critical stresses and the power-law distributions of crack waiting times. This explains that in materials containing long-range structural defects arrested microcracks are experimentally observable [7].

Crack tip equation of motion.—Let a single planar crack extend from $-x/2$ to $+x/2$ along the x direction of a 2D elastic medium of size L loaded in mode I by a uniform external stress σ. In a perfectly homogeneous elastic medium the Griffith crack energy is the sum of the elastic energy gain $E_{\mathrm{el}}(x)$ and the crack surface energy $E_s(x)$. The driving force for the crack tip advance is the release of the

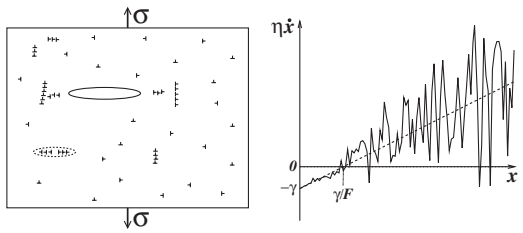

FIG. 1. Left: Sketch of an arrested crack (solid ellipse) in a random array of frozen dislocations (with cores represented by symbols ⊥); the dashed ellipse indicates a favorable region for crack nucleation. Right: Typical realization of the effective force $-\gamma + Fx + f_d(x)$; see Eq. (2a), acting on the crack tip for short-range correlated random forces $f_d(x)$ with $\delta = 3$.

elastic energy quantified by the *static energy release rate*. $G(x) = -\partial_x E_{\rm el} = \pi\sigma^2 x/2Y \equiv Fx$ [1], where $Y = E/(1-\nu^2)$ is the 2D Young's modulus. The energy release is balanced by the *specific fracture energy* γ, related to the crack surface energy E_s by $\gamma = \partial_x E_s$. Griffith's criterion for the onset of crack growth gives $G > G_c = \gamma$ [2], where G_c is the *critical energy release rate* that can be reached by increasing the crack length beyond the critical value $x_c = \gamma/F = 2\gamma Y/\pi\sigma^2$.

Material heterogeneities are described by the frozen random forces $f_d(x)$ which are included into the Griffith's force balance. We adapt Gaussian distributed random forces with zero mean value, $\overline{f_d(x)} = 0$, where the overbar denotes the average over disorder, and consider two types of forces: Short-range correlated forces (SRCF) $\overline{f_d(x)f_d(x')} = \Delta_\delta x^\delta \delta_a(x-x')$ (where a is a microscopic cutoff length) and long-range correlated forces (LRCF) $\overline{f_d(x)f_d(x')} = \Delta_\delta x^{(\delta-1)/2} x'^{(\delta-1)/2}$. Δ_δ is the strength of the random forces proportional to the defect concentration, and the exponent δ characterizes the elastic interaction between frozen defects and the crack. Note that for both types of random forces the corresponding potential energies $E_d(x)$ of the crack tip defined by the relation $f_d(x) = -\partial_x E_d$ have the same correlations $\overline{[E_d(x+z) - E_d(x)]^2} \sim \Delta_\delta z^{\delta+1}$ for $z \gg x$. It was shown in Ref. [5] that random bonds (random fracture toughness) result in SRCF with $\delta = 0$, impurities produce SRCF with $\delta = 1$, and dislocations induce LRCF with $\delta = 3$; i.e., δ increases with the range of the elastic interaction between the crack tip and defects [8].

Dissipation occurs mostly near the crack tip, where the elastic energy transforms into heat via plastic deformation [6], and can thus be described as the local viscous force exerted on the tip, $-\eta\dot{x}$, with η being the tip viscosity. Including the thermal force $\zeta(t)$ acting on the crack tip into the force balance, we obtain the overdamped equation for the crack tip motion as $\eta\dot{x} = G(x) - \gamma + f_d(x) + \zeta(t)$. As we focus on slow crack dynamics thermal fluctuations facilitate both transient crack healing and extension, and the system is close to thermal equilibrium such that it is justified to use $\langle\zeta(t)\rangle = 0$ and correlations $\langle\zeta(t)\zeta(t')\rangle = 2\eta T\delta(t-t')$ ($k_B \equiv 1$) for thermal forces. In order to complete the description of dynamics, the kinetic energy of the elastic medium has to be taken into account via the *dynamic energy release rate* $G(x,\dot{x})$ [3]:

$$G(x,\dot{x}) = A(\dot{x})G(x) \approx (1 - \dot{x}/c_R)G(x). \tag{1}$$

In general, $A(\dot{x})$ decreases monotonically with increasing crack tip velocity $\dot{x}$, starting with $A(0) = 1$ in the static limit and reaching zero $A(c_R) = 0$ at the Rayleigh wave velocity c_R [3]. The approximation in Eq. (1) complies with most experiments and will be used in what follows. In a homogeneous material the dynamic force balance $G(x,\dot{x}) = \gamma$ generalizes the Griffith's criterion and describes the energy flux into the crack tip and its subsequent conversion into crack surface energy. Including the viscous force $-\eta\dot{x}$, the thermal force $\zeta(t)$, and the frozen random forces $f_d(x)$ into the dynamic force balance, $-\eta\dot{x} + G(x,\dot{x}) = \gamma - f_d(x) - \zeta(t)$, one finally arrives at the following equation of motion:

$$\eta\dot{x} = B(x)[-\gamma + Fx + f_d(x) + \zeta(t)], \tag{2a}$$

$$B(x) \equiv [1 + (\gamma/\eta c_R)(x/x_c)]^{-1} \equiv [1 + x/b]^{-1}, \tag{2b}$$

which is a Langevin equation with multiplicative noise [9]; $b \equiv \eta c_R/F$ is the characteristic crack length. For small cracks $x \ll b$, $B(x) \approx 1$, and Eq. (2a) reduces to the usual overdamped dynamics. On the other hand, for large cracks, $x \gg b$, $B(x)$ vanishes as $B(x) \approx b/x$ giving rise to a low effective temperature.

Crack nucleation and propagation at $T = 0$.—In the absence of flaws [$f_d(x) = 0$], cracks can be thermally nucleated at temperatures $T > 0$ [10], but the material is stable against crack formation at $T = 0$. However, there is a nonzero probability that cracks of the critical size $x \sim x_c$ are ''nucleated'' by quenched disorder even at $T = 0$ if the typical disorder energy gain $\overline{E_d^2(x_c)} \sim \Delta_\delta x_c^{1+\delta}$, which scales in the same way for both SRCF and LRCF, compensates for the nucleation energy $\Delta E_c = \gamma^2/2F$ [5]. The resulting probability for disorder-induced nucleation,

$$p_{\rm nucl} = {\rm prob}[\Delta E_c + E_d(x_c) < 0] \sim e^{-\gamma^{3-\delta}F^{\delta-1}/\Delta_\delta}, \tag{3}$$

increases with the increasing disorder strength Δ_δ.

After its nucleation, a growing crack can get arrested by frozen disorder which, thus, can prevent fracture. Balancing the driving force and the typical random force developing over the distance x, $\overline{[f_d(x)^2]^{1/2}} \sim (\Delta_\delta/a)^{1/2}x^{\delta/2}$ for SRCF, we find a crack *arresting length* $x^* \equiv (F^2 a/\Delta_\delta)^{1/(\delta-2)}$ which characterizes the *static* force equilibrium (we focus on supercritical cracks $x \gg x_c$ and neglect the crack surface energy γ). For $\delta < 2$, the force equilibrium is unstable against further propagation giving rise to brittle fracture, whereas it is stable for $\delta > 2$ because sufficient driving forces become exponentially rare at $x > x^*$, which leads to a ductile fracture mechanism. For LRCF, we find an arresting length $x^* \equiv (F^2/\Delta_\delta)^{1/(\delta-3)}$ and the force equilibrium becomes stable for $\delta > 3$.

The probability $p_{\rm prop}$ for the crack propagation at $T = 0$ is given by the probability of finding a positive force on the crack tip for *all* $x > x_c$, $p_{\rm prop} \approx \prod_x {\rm prob}[Fx + f_d(x) > 0]$ [5]. SRCF show two qualitatively different behaviors depending on the value of δ:

$$p_{\rm prop} \sim \begin{cases} e^{-x^*/a} \sim e^{-(F_a/F)^{2/(2-\delta)}}, & \delta < 2 \\ [1 - e^{-(F/F_L)^2}]^{L/a}, & \delta \geq 2 \end{cases} \tag{4}$$

where $F_a \equiv (\Delta_\delta a^{\delta-3})^{1/2}$ and $F_L \equiv (\Delta_\delta L^{\delta-2}/a)^{1/2}$ are the characteristic forces, defined by $x^* = a$ and $x^* = L$, respectively. For $\delta < 2$ the fracture mechanism is brittle, as $p_{\rm prop}$ is independent of the system size L and strongly increases if x^* becomes comparable to the microscopic length a, whereas for $\delta \geq 2$, the fracture is ductile because

p_{prop} decreases for large L and becomes appreciable only if the stable crack of length x^* becomes comparable to L. For LRCF, on the other hand, we find

$$p_{\text{prop}} \sim 1 - e^{-(F/\tilde{F}_L)^2} \tag{5}$$

with $\tilde{F}_L \equiv (\Delta_\delta L^{\delta-3})^{1/2}$, which lacks an exponential dependence on L/a because the arresting forces are strongly correlated. The case of frozen dislocations (iii) with $\delta = 3$ is marginal. Cracks are arrested for disorders $\Delta_3 \gg F^2$ or up to high strain levels $\sigma/Y \ll b_{(\text{fd})} c_{(\text{fd})}^{-1/2} \sim 0.05$ where $b_{(\text{fd})}$ and $c_{(\text{fd})}$ are Burgers vector and 2D concentration of dislocations and we used an estimate for strong disorder in glass from Ref. [5]. Fracture will occur if a crack both nucleates and propagates with the probability $p_{\text{frac}} = p_{\text{nucl}} p_{\text{prop}}$. For SRCF with $\delta < 2$, i.e., random bonds (i) or random impurities (ii), the fracture probability is *finite* for large L. For LRCF with $\delta = 3$, i.e., frozen dislocations (iii), we find a *complementary* behavior $p_{\text{nucl}} \sim 1 - p_{\text{prop}}$, which enhances fracture toughness because at small defect densities the nucleation is unlikely, whereas at large defect concentrations the propagation is suppressed.

The fracture probability (as a function of F) equals the probability that F is larger than the critical force $F_c \equiv \max_x\{[\gamma - f_d(x)]/x\}$ of the given crack, i.e., $\text{prob}[F_c < F] = p_{\text{frac}}(F)$. For SRCF, cracks can nucleate at $N \sim (L/x_c)^2$ statistically independent seed locations in the sample [5] with critical forces drawn from the distribution $p_c(F_c) = \partial_F p_{\text{frac}}(F_c)$. Then the fracture occurs at the "weakest link" if the applied load F exceeds the *smallest* critical force of all N crack nuclei. The *average* fracture force $\bar{F}_{\text{frac}}$ follows from the condition $1 = N p_{\text{frac}}(\bar{F}_{\text{frac}})$. For $\delta < 2$, we find $\bar{F}_{\text{frac}} \sim F_a(\ln N)^{-(2-\delta)/2}$. Because $p_{\text{frac}}(F_c)$ and thus $p_c(F_c)$ decrease exponentially with $1/F_c$, see Eq. (4), the resulting *distribution* of fracture forces for $\delta < 2$ is an extreme value distribution of the Gumbel type,

$$\text{prob}\,[F_{\text{frac}} > F] \sim \exp[-c_1 N e^{-c_2 (\ln N)^{\delta/2} F_a/F}] \tag{6}$$

with constants c_1 and c_2. This result applies to random bonds (i) and random impurities (ii) and generalizes previous findings for random fuse models [11], which correspond to the special case of random bonds (i) with $\delta = 0$. The probability $\text{prob}[F_{\text{frac}} > F]$ in (6) equals the probability that the sample will not fracture and, thus, that all N statistically independent cracks are arrested. For LRCF, on the other hand, crack energies at different positions also have long-range correlations; thus extreme value statistics of fracture probabilities does not emerge and the fracture probability is simply given by $p_{\text{frac}} = p_{\text{nucl}} p_{\text{prop}}$.

Dynamics of thermally activated crack propagation.—Having established the conditions and probabilities for the crack arrest by heterogeneities at $T = 0$, we address the question of to what extent these findings have to be modified by thermal fluctuations. While at $T = 0$, any energy barrier leads to crack arrest, thermal fluctuations at $T > 0$ give rise to activated crack propagation. The equation of motion (2a) for the crack tip resembles those for the overdamped motion of a particle driven over the one-dimensional disorder potential $E_d(x)$ extensively reviewed in Ref. [12]. At low temperatures, the particle exhibits slow dynamics due to the wide distribution of energy barriers giving rise to anomalously slow diffusion, creep, or even particle arrest [13].

We consider an ensemble of cracks arrested at $T = 0$ by the random forces and address the question whether it stays arrested when the finite temperature, $T > 0$, is switched on. To this end we analyze the dynamics of a *typical* crack (since it is the typical crack that is arrested at $T = 0$, and propagating cracks represent *rare* events as follows from the previous section), which is described by the Fokker-Planck equation for the probability density $P(x, t)$ corresponding to the equation of motion (2a) [14]:

$$\partial_t P = -\partial_x J, \tag{7}$$

$$\eta J = -T\partial_x[B^2(x)P] + B(x)[-\gamma + Fx + f_d(x)]P. \tag{8}$$

After finding stationary solutions $P(x)$ for a nonzero constant current J and absorbing boundary conditions $P(L) = 0$, the normalization condition $\int_0^L dx P(x) = 1$ determines the fracture time $\tau_{\text{frac}} = 1/J$ [15],

$$\bar{\tau}_{\text{frac}} \approx \int_0^L dx \frac{\eta}{TB^2(x)} \int_0^\infty dz e^{-I(x,z)},$$

$$\text{with } I(x,z) \equiv \int_x^{x+z} du[Fu/TB(u) - \Delta_\delta u^\delta/T^2B^2(u)], \tag{9}$$

where we took the limit of infinite L and averaged over disorder. The behavior of $I(x, z)$ for large z governs the fracture time and is identical for both SRCF and LRCF [14]. Using the asymptotics for large u, $B(u) \approx b/u$, see Eq. (2b), we find a finite mean fracture time $\bar{\tau}_{\text{frac}}$ for $\delta < 0$, whereas it diverges for $\delta > 0$. For $\delta = 0$, the mean fracture time is infinite for $\Delta_0 > \Delta_{0,c} \equiv FTb = T\eta c_R$, i.e., above the threshold disorder strength $\Delta_{0,c}$, which is independent of the driving force F.

Now we derive the distribution of random energy barriers that govern the activated dynamics [16]. As follows from the equation of motion (2a), the effective random energy controlling thermal activation is $\phi(x)$ with $\partial_x \phi \equiv [-Fx - f_d(x)]/B(x)$. Therefore, we have to find the distribution of barriers $p(E)$ developing in the random energy landscape $\phi(x)$ of a particle located initially at $x = x_i$. This distribution evolves from the Gaussian distribution of random forces $f_d(x)$ and can be written as path integral in the "energy space" applying the formalism that has been developed in Ref. [16]. After some algebra, we finally find the effective barrier distribution

$$p(E) \sim e^{-\text{const}\times(E/E_0)^{1-\delta/3}} \tag{10}$$

for large cracks ($x \gg x_i$ and $x \gg b$) both for SRCF and LRCF, where $E_0 = b^{-1}\Delta_\delta^{3/(3-\delta)} F^{-(3+\delta)/(3-\delta)}$ is the char-

acteristic barrier energy. The distribution (10) attains a simple exponential form for $\delta = 0$. For $\delta > 3$ large barriers are no longer rare, and $p(E)$ cannot be normalized. Consider the crack tip starting at x_i and traveling over a distance $x_t \gg x_i$ for the time t. For $0 \leq \delta < 3$, the dynamics is controlled by the highest barrier, E_t, it meets, obtained from the condition $1 = (x_t/a) \int_{E_t}^{\infty} dE p(E)$. Then $E_t \sim E_0 \ln^{3/(3-\delta)} x_t$, and using the Arrhenius relation $t \sim e^{E_t/T}$ we find

$$x_t \sim \exp[(T/E_0)^{1-\delta/3} \ln^{1-\delta/3} t] \quad \text{for } 0 \leq \delta < 3. \quad (11)$$

For $\delta = 0$, i.e., random bonds (i), this represents anomalously slow diffusion with the power-law dynamics $x_t \sim t^{T/E_0}$ where $E_0 = \Delta_0/Fb$. For $T < E_0$ or above the threshold disorder strength $\Delta_{0,c}$, we find the vanishing mean velocity $x_t/t \approx 0$ in agreement with our above result of a diverging mean fracture time $\bar{\tau}_{\text{frac}}$ for $\delta = 0$; $\delta < 0$ leads to fast brittle fracture $x_t \sim t$ agreeing with our conclusion about the corresponding *finite* mean fracture time. For $\delta > 0$ the crack motion law is slower than any power with $x_t/t \approx 0$, representing the effective crack arrest (meaning an infinite mean fracture time). For $\delta \geq 3$, the complete crack arrest, $x_t \approx 0$, occurs. Thus an ensemble of cracks that was arrested at $T = 0$ remains effectively arrested (in the sense of an infinite fracture time or zero average velocity) for heterogeneities with $\delta > 0$, which include frozen dislocations (iii) and random impurities (ii). For random bonds (i) with $\delta = 0$ we find the anomalously slow diffusion with the power-law dynamics.

Conclusion and discussion.—We have derived the equation of motion (2a) for the crack tip by incorporating effects from dissipation, thermal fluctuations, and frozen heterogeneities into the dynamic fracture criterion $G(x, \dot{x}) = \gamma$. The tip equation of motion is an overdamped Langevin-type equation for a particle in a one-dimensional disordered potential. We have obtained the conditions and probabilities for crack arrest at zero temperature as a function of the applied stress F and the type of heterogeneities involved as described by the exponent δ [8] both for the short-range and long-range correlated forces. For SRCF, we find complete crack arrest for $\delta \geq 2$ and the extreme value probability (6) of crack arrest for $\delta < 2$. For LRCF, cracks get arrested for $\delta > 3$, and the fracture probability is drastically reduced for hard-worked materials [17] containing frozen dislocations (iii) ($\delta = 3$). Cracks that are arrested at $T = 0$ can propagate by thermal activation at finite-temperatures $T > 0$. For heterogeneities with $\delta \geq 0$, i.e., also for random fracture toughness (i) and impurities (ii), the disorder potential leads to the slow crack dynamics (11) with zero mean velocity as the crack tip gets trapped in the deep potential minima. This trapping mechanism is much more efficient than the crack capture by crystal lattice effects [18] and explains the existence of arrested metastable *microcracks* in heterogeneous materials with sizes that can be considerably larger than the critical crack length of the homogeneous material; this effect has been observed in a number of recent experiments [7]. Experimentally observed fracture precursors in heterogeneous materials with power-law waiting time distributions [7,19,20] can also be explained in the framework of our theory as characteristics of the case $\delta = 0$ of random fracture toughness (i). It remains an open question for future investigations whether the ensembles of arrested microcracks become unstable with respect to microcrack coalescence and slow crack growth by cyclic loading in fatigue experiments.

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